Math 10A
Worksheet, Discussion #20; **Friday, 7/13/2018**
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1 Error Bounds

1.1 Concepts

1. The formula for the errors of integrating $\int_a^b f(x)dx$ are:

$$E_L = E_R = \frac{K_1(b-a)^2}{2n}, \quad E_T = \frac{K_2(b-a)^3}{12n^2}, \quad E_M = \frac{K_2(b-a)^3}{24n^2}, \quad E_S = \frac{K_4(b-a)^5}{180n^4},$$

where K_i is the maximum $|f^(i)(x)|$, the ith derivative of f, is on the interval $[a,b]$.

1.2 Problems

2. True **FALSE** For calculating the error bound when using left endpoint method when approximating the integral of f on the interval $[a,b]$, we use $K_1 = f'(a)$.

> **Solution:** We define K_1 to be the maximum of $f'(x)$ on the interval $[a,b]$. This may occur at a but that is not necessary.

3. True **FALSE** The error bound gives us what the exact error of using the different approximation techniques are.

> **Solution:** The error bounds, as their name suggests, just allow us to bound the error. The actual error may be less than the bound (or even 0 as seen in question 1).

4. True **FALSE** If the second derivative is negative, then the Trapezoid rule and midpoint rule both underestimate the true area.

> **Solution:** The Trapezoid rule will underestimate the area while the midpoint rule will overestimate it.

5. True **FALSE** If the first derivative is positive, then the left endpoint and right endpoint method both underestimate the true area.

Solution: The left endpoint method would underestimate the area and the right endpoint would overestimate it (think about $y = x$).

6. How many intervals do we need to use to approximate $\int_1^2 x^2 dx$ within $0.001 = 10^{-3}$ using the midpoint rule? Trapezoid rule? Simpson's rule?

Solution: We take the error bound equation, set the error to be our desired bound, and solve for n. So for example, for midpoint rule, we have that $K_2 = \max|22|$ on the interval $[1, 2]$, which is just 2 so $K_2 = 4$ and we have

$$E_M = 10^{-3} = \frac{K_2(b-a)^3}{24N^2} = \frac{2}{24N^2} \implies N = \sqrt{\frac{2000}{24}} = 9.128.$$

When we are asking for the minimal number of intervals, we need an integral number and hence we take the ceiling 10 because anything greater than 9.128 gives us a good bound, and 9 does not.

The table is shown below:

Error	E_L	E_R	E_M	E_T	E_S
0.01	201	201	4	5	2
0.001	2001	2001	10	14	2
0.0001	20001	20001	30	42	2

7. How many intervals do we need to use to approximate $\int_0^1 \cos(2x)dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We have $f'(x) = -2\sin(2x)$ so $K_1 = 2$, $f''(x) = -4\cos(2x)$ so $K_2 = 4$, and $K_4 = 16$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	101	101	5	7	4
0.001	1001	1001	14	19	4
0.0001	10001	10001	42	59	6

8. How many intervals do we need to use to approximate $\int_0^2 e^{2x} dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We have $f'(x) = 2e^{2x}$ so $K_1 = 2e^4$ and $f''(x) = 4e^{2x}$ so $K_2 = 4e^4$ and $K_4 = 16e^4$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	21840	21840	86	122	12
0.001	218394	218394	271	383	22
0.0001	2183927	2183927	854	1208	36

9. How many intervals do we need to use to approximate $\int_{-1}^{1} x^3 dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = 3, K_2 = 6, K_4 = 0$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	601	601	15	21	2
0.001	6001	6001	46	64	2
0.0001	60001	60001	142	201	2

10. How many intervals do we need to use to approximate $\int_{1}^{3} \ln x dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = K_2 = 1$ and $K_4 = 6$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	201	201	7	9	4
0.001	2001	2001	19	27	8
0.0001	20001	20001	59	83	12

11. How many intervals do we need to use to approximate $\int_{1}^{2} xe^x dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = 3e^2, K_2 = 4e^2, K_4 = 6e^2$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	1109	1109	12	17	4
0.001	11085	11085	36	51	6
0.0001	110837	110837	112	158	8

12. How many intervals do we need to use to approximate $\int_1^4 \sqrt{x}dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = \frac{1}{2}, K_2 = \frac{1}{4}, K_4 = \frac{15}{16}$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	226	226	6	9	4
0.001	2251	2251	18	25	8
0.0001	22501	22501	54	76	12